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Examples on Similarity in L^p

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Let Q be a polynomial with real coefficients on R such that convolution by $\exp(iQ)$ defines a bounded operator H on L^p for some p $(1 < p < \infty)$. Then the perturbations $M + cH$ are similar to M on L^p, where M is the multiplication by x operator with maximal domain.

INTRODUCTION

Let $C_c^\infty = C_c^\infty(R)$ denote the space of all infinitely differentiable complex functions on the real line R with compact support. If Q is a polynomial with real coefficients, consider the convolution operator $Hf = \exp(iQ) * f$ on C_c^∞. We say that Q is p-admissible (where $1 < p < \infty$) if H extends as a bounded operator on $L^p = L^p(R)$. For example, the polynomial $Q(x) = ax^2$ is known to be 2-admissible for non-zero real a.

Let M denote the multiplication operator $(Mf)(x) = xf(x)$ with maximal domain in L^p: $D(M) = \{f \in L^p; Mf \in L^p\}$.

THEOREM. *If Q is a p-admissible for some p $(1 < p < \infty)$, then $M + cH$ is similar to M on L^p for any scalar c.*

PROOF OF RESULT

We obtain the theorem as an application of the following (cf. [1, Corollary 12.3]): let V, C be commuting bounded operators in a Banach space, and let S be a closed operator with V-invariant domain, such that $[S, V] \subset C$. Then for any function g holomorphic on the spectrum of V, the operator $S + g'(V)\,C$ is similar to S.

With $S = M$ and $C = H$, we need an operator $V = K$ satisfying the above relations with M and H; then the more general statement "$M + g'(K)\,H$ is similar to M" will follow (the special case $g(z) = cz$ gives our theorem).

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For $r > 0$, let

$$k_r(x) = \begin{cases} x^{-1} e^{iQ(x)}, & |x| \geqslant r \\ 0 & |x| < r, \end{cases}$$

and let K_r be the corresponding convolution operator on C_c^∞.

Fix p ($1 < p < \infty$). A special case of a result of S. Chanillo and M. Christ (Weak (1, 1) bounds for oscillatory singular integrals, *Duke Math. J.* **55** (1987), 141–155; also F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals, *J. Funct. Anal.* **73** (1987), 179–194) establishes that for each $f \in C_c^\infty$, $K_r f$ converges pointwise almost everywhere and in L^p (as $r \to 0$) to an element $Kf \in L^p$; the operator K on C_c^∞ is bounded with respect to the L^p-norm, and extends therefore to a bounded operator on L^p.

Although "usual" convolution operators commute, the needed commutativity of H and K needs verification (for a p-admissible polynomial Q).

Let $h_r = Mk_r$ and $H_r f = Hf - h_r * f$ for $f \in C_c^\infty$. If Q is p-admissible, it follows from the lemma on p. 187 of the cited paper of Ricci and Stein, that there exists a constant C_p independent of r and Q, such that

$$\|K_r\| \leqslant C_p(1 + \|K\|) \tag{1}$$

and a similar inequality for H_r.

Consider the sequence $\{H_n; n = 1, 2, ...\}$. By (1) for H_n, the sequence $\{H_n f\}$ is bounded in L^p for $f \in C_c^\infty$ fixed. If g is a weak limit point, the Dominated Convergence and Fubini Theorems imply easily that $g = Hf$, and therefore $H_n f \to Hf$ weakly in L^p. Note that $H_n f = h_n^\# * f$, where $h_n^\#(x) = e^{iQ(x)}$ for $|x| < n$ and vanishes for $|x| \geqslant n$ (so $h_n^\# \in L^1$); since $k_r \in L^2$, commutativity and associativity of convolution between L^1 and L^2 functions with at most one L^2 function involved; cf. N. Dunford and J. T. Schwartz, "Linear Operators," p. 951, Vol. II, Interscience, New York, 1962) imply that

$$\begin{aligned} K_r H_n f &= k_r * (h_n^\# * f) = (k_r * h_n^\#) * f = f * (k_r * h_n^\#) \\ &= (f * k_r) * h_n^\# = h_n^\# * (k_r * f) = H_n K_r f. \end{aligned}$$

Keep $f \in C_c^\infty$ and $r > 0$ fixed. Since $k_r \in L^q$ ($q = p/(p-1)$) and $H_n f \to Hf$ weakly in L^p, $K_r H_r f = k_r * H_n f \xrightarrow[n \to \infty]{} k_r * Hf = K_r Hf$ pointwise. By (1) for H_n, we have for any $g \in C_c^\infty$,

$$\begin{aligned} |(K_r H_n f)(x - y)\, g(y)| &\leqslant \|k_r\|_q \, \|H_n f\|_p \, |g(y)| \\ &\leqslant C_p \, \|k_r\|_q \, (\|H\| + 1) \, \|f\|_p \, |g(y)|. \end{aligned}$$

By Dominated Convergence, it follows that

$$(K_r H_n f) * g \xrightarrow[n \to \infty]{} (K_r Hf) * g \qquad \text{pointwise.}$$

On the other hand, $H_n(K_r f) \xrightarrow[n \to \infty]{} H(K_r f)$ weakly, and therefore $(H_n K_r f) * g \to_{n \to \infty} (HK_r f) * g$ pointwise. Since $g \in C_c^\infty$ is arbitrary, it follows that $K_r Hf = HK_r f$, and since H is a bounded operator on L^p, we conclude that $KHf = HKf$ by letting $r \to 0$. Thus $KH = HK$ by density of C_c^∞ in L^p.

Clearly, $C_c^\infty \subset D(M)$. Writing $x = (x - y) + y$, we see that for $f \in C_c^\infty$,

$$(MK_r f)(x) = \int_{|x-y| \geq r} e^{iQ(x-y)} f(y)\, dy + (K_r Mf)(x).$$

When $r \to 0$, the first term on the right converges pointwise to $Hf(x)$, and the second term converges a.e. to $(KMf)(x)$. Since the left hand side converges a.e. to $(MKf)(x)$, we conclude that for $f \in C_c^\infty$,

$$MKf = KMf + Hf \qquad \text{a.e.} \tag{2}$$

Since the right hand side of (2) is in L^p, we see in particular that $Kf \in D(M)$ for all $f \in C_c^\infty$. Let $f \in D(M)$. We may choose $f_j \in C_c^\infty$ such that $f_j \to f$ in L^p and almost everywhere, and such that $|f_j| \leq |f|$ a.e. Then $Mf_j \to Mf$ a.e. and $|Mf_j| \leq |Mf| \in L^p$, so that $Mf_j \to Mf$ in L^p by Dominated Convergence. Now $Kf_j \in D(M)$, $Kf_j \to Kf$ in L^p, and by (2),

$$M(Kf_j) \to K(Mf) + Hf \qquad \text{in } L^p.$$

Since M is closed, it follows that $Kf \in D(M)$ and $M(Kf) = K(Mf) + Hf$, i.e., $KD(M) \subset D(M)$ and $[M, K] = H$ on $D(M)$. The result now follows as observed at the beginning of the proof.

Reference

1. S. Kantorovitz, Spectral theory of Banach space operators, *in* "Lecture Notes in Mathematics," Vol. 1012, Springer, New York, 1983.